Graphs for Cone Preserving Maps*

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ABSTRACT

Let K be a closed, pointed, full cone in a finite dimensional real vector space. We associate with a linear map A for which $AK\subseteq K$ four directed graphs. For two of the graphs the vertex set is the collection of all faces of K, and for two the vertices are all the extreme rays of K. We relate the irreducibility and primitivity of A to the strong connectedness of some of these graphs.

I. INTRODUCTION

Richard Varga [4] associated a directed graph with a nonnegative matrix and applied this concept to numerical procedures. A survey of recent developments in this area can be found in [2]. Here we associate directed graphs with a cone preserving map and characterize irreducibility and primitivity in terms of two of these graphs.

Let V be a real vector space of dimension d. We shall consider a closed, full, pointed cone K in V. That is, $K\subset V$ satisfies

(1) if $x,y\in K$, $\alpha,\beta\geqslant 0$, then $\alpha x+\beta y\in K$,
(2) K is closed in the natural topology of V,
(3) int $K\neq\varnothing$ (or equivalently, span $K=V$),
(4) $K\cap(-K)=\{0\}$.

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If $x \in K$ we write $x \geqslant 0$. If $x \in \operatorname{int} K$, we write $x \gg 0$, while $x > 0$ means $x \geqslant 0$ and $x \neq 0$. The set $\Pi(K) = \{A \in \operatorname{Hom} V\colon AK \subseteq K\}$ is easily seen to be a closed, full, pointed cone in $\operatorname{Hom} V$, and the notations $A \geqslant 0$ et cetera have the obvious meanings with respect to $\Pi(K)$.

DEFINITION 1. A *face* F is a subcone of K such that

$$0 \leqslant y \leqslant x \text{ and } x \in F \quad \text{imply} \quad y \in F.$$

The set of all faces is denoted by $\mathcal{F}$. An *extreme ray* is a one dimensional face of K. The set of all extreme rays is denoted by $\mathcal{E}$.

REMARK 1. If $S \subset K$, then the set

$$\Phi(S) = \bigcap \{F\colon S \subseteq F, F \text{ a face of } K\}$$

is a face called the face generated by S. If $S = \{x\}$, we write $\Phi(x)$ for simplicity.

NOTATION. If $F \in \mathcal{F}$ we also write $F \lhd K$. If $F, G \in \mathcal{F}$ and $F \subseteq G$, we write $F \lhd G \lhd K$. This notation is easily checked to be consistent, that is, F is a face of G where G is considered to be a cone in its span (cf. [1]).

DEFINITION 2. A vector $x \in K$ is called an extremal if $\Phi(x) \in \mathcal{E}$.

REMARK 2. If x is extremal and if $0 \leqslant y \leqslant x$, then $\alpha y = x$ for some $\alpha \geqslant 0$.

If $A \in \Pi(K)$, we may associate with A four directed graphs. If $F, G \in \mathcal{F}$, we say there is an $\mathcal{G}$-edge from F to G, $\mathcal{G}(F, G)$, iff $G \lhd \Phi[(I + A)F]$. We say there is a $\mathcal{P}$-edge from F to G, $\mathcal{P}(F, G)$, iff $G \lhd \Phi[AF]$. Note that every $\mathcal{P}$-edge corresponds to an $\mathcal{G}$-edge, since $\Phi[AF] \lhd \Phi[(I + A)F]$. Let $\mathcal{G}$ and $\mathcal{P}$ denote the sets of $\mathcal{G}$-edges and $\mathcal{P}$-edges. Then $(\mathcal{F}, \mathcal{G})$ denotes the directed graph with $\mathcal{F}$ as the set of vertices and $\mathcal{G}$ as the set of edges. The other pairings are defined analogously. It is easily checked that the following inclusion diagram holds, where $(\mathcal{E}, \mathcal{G})$ is a subgraph of $(\mathcal{F}, \mathcal{G})$ and so on:

$$\begin{tikzcd}
 & (\mathcal{F}, \mathcal{G}) \arrow[dl, dash] \arrow[dr, dash] & \\
(\mathcal{F}, \mathcal{P}) \arrow[dr, dash] & & (\mathcal{E}, \mathcal{G}) \arrow[dl, dash] \\
 & (\mathcal{E}, \mathcal{P}) &
\end{tikzcd}$$

$(\mathcal{E}, \mathcal{P})$ and $(\mathcal{F}, \mathcal{P})$ are sometimes called partial subgraphs, since in general $\mathcal{P} \neq \mathcal{G}$.

DEFINITION 3.

(a) $A \in \Pi(K)$ is *irreducible* iff A leaves no nontrivial face of K invariant.

(b) $A \in \Pi(K)$ is *primitive* iff for all $x > 0$ there is a positive integer k such that $A^k x \gg 0$.

REMARK 3. A is primitive iff there is an integer $k > 0$ such that for all $x > 0$, $A^k x \gg 0$ [3].

Recall that a directed graph G is strongly connected iff for any two vertices v_1, v_2 there is a directed path from v_1 to v_2. Since $K \in \mathcal{F}$, this assumption is too strong. Instead we use a slightly modified definition of strong connectivity.

DEFINITION 4. We say that $(\mathcal{F}, \mathcal{G})$ [respectively $(\mathcal{F}, \mathcal{P})$] is *strongly connected* iff for any two nonzero proper faces F, G there is a path of $\mathcal{G}$-edges [respectively $\mathcal{P}$-edges] from F to G.

REMARK 4. Let $F, G \in \mathcal{F}$. There is a path of $\mathcal{P}$-edges (respectively $\mathcal{G}$-edges) of length k from F to G iff $G \triangleleft \Phi[A^k F]$ (respectively $G \triangleleft \Phi[(I+A)^k F]$).

II. MAIN RESULTS

THEOREM 1. *A is irreducible iff $(\mathcal{F}, \mathcal{G})$ is strongly connected.*

Proof. Let A be irreducible, and let F, G be nonzero proper faces of K. Vandergraft [3] showed that the irreducibility of A is equivalent to $(I+A)^{n-1} \gg 0$. Hence $K = \Phi[(I+A)^{n-1} F] \triangleleft G$, and by Remark 4, there is a path of $\mathcal{G}$-edges from F to G. Conversely, suppose A is reducible. Let F be a nonzero proper invariant face of A, and let $G = \Phi(x)$, when x is an extremal not in F. Then $\Phi[(I+A)F] = F$ and no path can lead from F to G. Thus $(\mathcal{F}, \mathcal{G})$ is not strongly connected. ∎

Minor modification of the proof of the second half of the theorem establishes the following proposition.

PROPOSITION 1. *If $(\mathcal{F}, \mathcal{G})$ is strongly connected, then A is irreducible.*

The converse does not hold. To see this, let K be the proper polyhedral cone in $\mathbb{R}^4$ generated by the extreme vectors

$$a_1=\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\quad a_2\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\quad a_3=\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\quad a_4=\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\quad a_5=\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}.$$

Let $A\in\operatorname{Hom}\mathbb{R}^4$ be the projection $(x_1,x_2,x_3,x_4)^T\mapsto(x_1,x_2,x_3,0)^T$ followed by the linear mapping on the range of the projection which is determined by $a_1\mapsto a_1+a_2$, $a_2\mapsto a_2+a_3$, and $a_3\mapsto a_3+a_1$. Obviously, $AK\subset\operatorname{span}\{a_1,a_2,a_3\}\cap K$. In fact A is primitive, since $A^2\gg 0$. On the other hand we have

$$(A+1)a_1=2a_1+a_2,\qquad (A+1)a_2=2a_2+a_3,\qquad (A+1)a_3=2a_3+a_1.$$

Thus in the directed graph $(\mathcal{E},\mathcal{G})$ there are edges from a_1 to a_2, a_2 to a_3, and a_3 to a_1. There is no path from, say, a_1 to a_4. Thus $(\mathcal{E},\mathcal{G})$ is not strongly connected. We shall show in Theorem 2 that A is primitive iff $(\mathcal{F},\mathcal{P})$ is strongly connected. Assuming, this we note that $(\mathcal{F},\mathcal{P})$ strongly connected does not imply $(\mathcal{E},\mathcal{G})$ strongly connected.

Clearly if A is primitive, then $(\mathcal{F},\mathcal{P})$ is strongly connected. Thus we need to establish one direction in the next result.

THEOREM 2. *Let $A\in\Pi(K)$ and $\dim V>2$. Then A is primitive iff $(\mathcal{F},\mathcal{P})$ is strongly connected.*

We shall use two lemmas to establish the nontrivial direction of Theorem 2.

LEMMA 1. *Let $A\in\Pi(K)$, and let $(\mathcal{F},\mathcal{P})$ be strongly connected. If there is an $x\in\partial K$ such that $A^p x\gg 0$ for some p, then A is primitive.*

Proof. Let $y>0$. Since $(\mathcal{F},\mathcal{P})$ is strongly connected, there is a path from $\Phi(y)$ to $\Phi(x)$. By Remark 4, $\Phi[A^k y]\lhd\Phi(x)$ for some positive integer k, whence

$$K=\Phi(A^p x)\lhd\Phi(A^{k+p}).$$

Thus $A^{k+p}y\gg 0$.

LEMMA 2. *Let* $A \in \Pi(K)$, $K=$index of A. *Then* A *is irreducible iff* $\operatorname{Im} A^k \cap K \not\subseteq \partial K$ *and* $A|_{\operatorname{Im} A^k}$ *is irreducible with respect to* $\operatorname{Im} A^k \cap K$. *Further,* A *is primitive iff* $\operatorname{Im} A^k \cap K \not\subseteq \partial K$ *and* $A|_{\operatorname{Im} A^k}$ *is primitive with respect to* $\operatorname{Im} A^k \cap K$.

We leave the proof to the reader. The important point is: if $\operatorname{Im} A^k \cap K \subseteq \partial K$, then $\Phi(\operatorname{Im} A^k \cap K)$ is a proper face of K; if $\operatorname{Im} A^k \cap K \not\subseteq \partial K$, then (relative interior of $\operatorname{Im} A^k \cap K) \subseteq \operatorname{int} K$ and $\operatorname{rel\,bdy}(\operatorname{Im} A^k \cap K) \subseteq \partial K$.

Proof of Theorem 2. We assume that $(\mathcal{F}, \mathcal{P})$ is strongly connected but that A is not primitive. By Lemma 1 $A(\partial K) \subseteq \partial K$. Further A is irreducible. Let $K=$index of A. By Lemma 2, $A|_{\operatorname{Im} A^k}$ is irreducible on $\operatorname{Im} A^k \cap K$, the relative interior of $\operatorname{Im} A^k \cap K$ is contained in $\operatorname{int} K$, and $\operatorname{rel\,bdy}(\operatorname{Im} A^k \cap K) \subseteq \partial K$. Thus

$$A|_{\operatorname{Im} A^k}\big(\operatorname{rel\,bdy}(\operatorname{Im} A^k \cap K)\big) \subseteq \operatorname{rel\,bdy}(\operatorname{Im} A^k \cap K).$$

Since $A|_{\operatorname{Im} A^k}$ is nonsingular, $A|_{\operatorname{Im} A^k} \in \operatorname{Aut}(\operatorname{Im} A^k \cap K)$. [Recall that $A \in \operatorname{Aut}(K)$ iff $A^{-1} \in \Pi(K)$.] Thus A sends extreme rays of $\operatorname{Im} A^k \cap K$ to extreme rays of $\operatorname{Im} A^k \cap K$. To see this suppose x and y are distinct extremals for which $y \leqslant_{\operatorname{Im} A^k \cap K} Ax$. Then $z=(A|_{\operatorname{Im} A^k})^{-1} y \leqslant_{\operatorname{Im} A^k \cap K} x$, whence $z=\alpha x$ for some $\alpha>0$. Thus $y=\alpha Ax$. But since $(\mathcal{F}, \mathcal{P})$ is strongly connected, if F is a maximal face of $\operatorname{Im} A^k \cap K$ and if x determines an extreme ray of $\operatorname{Im} A^k \cap K$, there is a path of $\mathcal{P}$-edges from $\Phi(x)$ to $\Phi(F)$. Thus each extreme ray of $\operatorname{Im} A^k \cap K$ is also a maximal face. We consider two cases.

Case 1. $\dim(\operatorname{Im} A^k \cap K)=2$. Let x_1 and x_2 be distinct extremals of $\operatorname{Im} A^k \cap K$. Then $\Phi(x_1)$ and $\Phi(x_2)$ are contained in maximal faces F_1 and F_2 of K. Since $\dim K > 2$, there is a $y \in \partial K$ such that $y \notin F_1 \cup F_2$. But $A\Phi(x_1) \subset \Phi(x_2) \lhd F_2$ and $A\Phi(x_2) \subset \Phi(x_1) \lhd F_1$. Thus there is no path from (say) $\Phi(x_1)$ to $\Phi(y)$ which contradicts the strong connectedness of $(\mathcal{F}, \mathcal{P})$.

Case 2. $\dim(\operatorname{Im} A^k \cap K)>2$. Then $\operatorname{Im} A^k \cap K$ is strictly convex, whence it has uncountably many extreme rays. But using paths of $\mathcal{P}$-edges, we may connect a fixed extreme ray of $\operatorname{Im} A^k \cap K$ to only countably many other extreme rays. This contradicts the fact that $(\mathcal{F}, \mathcal{P})$ is strongly connected. Hence A is primitive. ■

The dimension restriction in the hypothesis of Theorem 2 is needed. Let K be the usual positive orthant in $\mathbb{R}^2$, and let

$$A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then A is certainly imprimitive, but $(\mathcal{F}, \mathcal{P})$, which coincides with $(\mathcal{E}, \mathcal{P})$, is strongly connected.

Also in general the strong connectedness of $(\mathcal{E}, \mathcal{P})$ does not imply that of $(\mathcal{F}, \mathcal{P})$. Let K be the cone of elementwise nonnegative vectors in $\mathbb{R}^n$, $n \geqslant 3$, and let A be an irreducible but imprimitive matrix. That $(\mathcal{E}, \mathcal{P})$ is strongly connected follows from the usual proof that $(I+A)^m \gg 0$ implies the usual graph of A is strongly connected (cf. [4] or [2]). But $(\mathcal{F}, \mathcal{P})$ is clearly not strongly connected.

The implications of the strong connectedness of $(\mathcal{E}, \mathcal{P})$ are unclear. By suitable modifying the proof of Lemma 1 we can readily establish a modest result.

PROPOSITION 2. *Suppose $(\mathcal{E}, \mathcal{P})$ is strongly connected. Then A is primitive iff for some $x \in \operatorname{Ext} K$, $A^l x \gg 0$ for some positive integer l.*

Also as a corollary of Theorem 2, we have that if K is a strictly convex cone of dimension $\geqslant 3$, then $(\mathcal{E}, \mathcal{P})$ is strongly connected iff A is primitive.

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